

# Exactly solvable two-level quantum systems and Landau-Zener interferometry

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I present a simple algorithm based on a type of partial reverse-engineering that generates an unlimited number of exact analytical solutions to the Schrödinger equation for a general time-dependent two-level Hamiltonian. I demonstrate this method by deriving new exact solutions corresponding to fast control pulses that contain arbitrarily many tunable parameters. It is shown that the formalism is naturally suited to generating analytical control protocols that perform precise non-adiabatic rapid passage and Landau-Zener interferometry near the quantum speed limit. A general, exact formula for Landau-Zener interference patterns is derived.

Although they have pervaded quantum physics since its inception, very few time-dependent two-level quantum systems are known to be analytically solvable. Among the most famous examples of exactly soluble two-level evolution is the Landau-Zener-Stückelberg-Majorana<sup>1-5</sup> (LZSM) problem, which remains a very active area of research due to numerous applications pertaining to quantum phase transitions,<sup>6</sup> quantum control<sup>7-11</sup> and quantum state preparation.<sup>12-14</sup> The hyperbolic secant pulse of Rosen and Zener comprises another well known exact solution discovered in the same period.<sup>15</sup> This solution has played an important role in the contexts of self-induced transparency<sup>16</sup> and qubit control,<sup>17-19</sup> and a number of authors have since found that this example belongs to a larger family of analytical controls.<sup>20-30</sup> In the last few years, several of these examples have proven very beneficial to the fields of quantum control and quantum computation,<sup>17,18,31-34</sup> where analytical solutions are often central in the design of control fields that are fast, precise, and robust against noise. However, the rarity of exactly solvable two-state systems has severely limited one's options in developing an analytical approach to qubit gate design.

In a recent work,<sup>35</sup> a systematic method for deriving arbitrarily many families of exactly solvable two-state systems was presented, vastly extending the number of known analytical solutions. This method allows one to input many of the basic features of the desired control field and then compute exactly the corresponding evolution of the system with the provided formulas. However, a limitation of this work is that it applies only to systems where the driving is along a single axis of the Bloch sphere, such as in the case of electrically-driven singlet-triplet qubits,<sup>36-39</sup> making it inapplicable to the majority of driven two-level systems.

In this paper, I address this limitation by presenting a method to generate arbitrarily many families of solutions in the most general case where the two-level Hamiltonian has time-dependence along any set of axes of the Bloch sphere. Of course, one can easily generate exactly solvable Hamiltonians by first choosing the evolution operator and then differentiating to obtain the corresponding Hamiltonian, but it is challenging to arrive at a physically meaningful Hamiltonian in this way. In contrast,

the method presented here allows one to specify the basic form and many of the features of the Hamiltonian whose evolution one wishes to solve before proceeding to compute the exact solution for this evolution. This method for generating exact solutions has important applications in a vast range of quantum physics problems, including the development of quantum controls for essentially any quantum computing platform and control protocols for performing LZSM interferometry and non-adiabatic rapid passage (NARP). I illustrate this by deriving new, exactly solvable LZSM driving fields and control pulses that execute a desired evolution at speeds approaching the quantum speed limit<sup>40-44</sup> (QSL). Attaining fast evolution times is especially crucial in quantum computing where quantum gates need to be performed on timescales much shorter than the decoherence time. In the case of periodic driving through a level anti-crossing, I show that the formalism allows one to easily derive analytical expressions for LZSM interference patterns and conditions for coherent destruction of tunneling.<sup>45,46</sup>

The Hamiltonian we will consider has the general form

$$H = b(t) \cdot \sigma, \quad (1)$$

where  $b(t) = [b_x(t), b_y(t), b_z(t)]$  is a three-vector whose components are arbitrary real functions, and  $\sigma = [\sigma_x, \sigma_y, \sigma_z]$  is a vector of Pauli matrices. This Hamiltonian describes any time-dependent two-level system. The components of the effective magnetic field  $b(t)$  can each admit various physical interpretations, most commonly as either driving fields or time-dependent energy splittings. The evolution operator for this Hamiltonian obeys a Schrödinger equation whose form can be made compact by first parametrizing the evolution operator as

$$U = \begin{pmatrix} u_{11} & -u_{21}^* \\ u_{21} & u_{11}^* \end{pmatrix}, \quad |u_{11}|^2 + |u_{21}|^2 = 1, \quad (2)$$

and then transforming to a rotating frame:  $v_{11} = e^{i \int_0^t dt' b_z(t')} u_{11}$ ,  $v_{21} = e^{-i \int_0^t dt' b_z(t')} u_{21}$ . Writing  $b_x = \beta(t) \cos \varphi(t)$  and  $b_y = \beta(t) \sin \varphi(t)$ , and defining  $\alpha(t) \equiv 2 \int_0^t dt' b_z(t') - \varphi(t)$ , we have

$$v_{11} = -i\beta e^{i\alpha} v_{21}, \quad v_{21} = -i\beta e^{-i\alpha} v_{11}. \quad (3)$$

Here, it is manifest that the evolution operator in the rotating frame depends on only two real functions,  $\alpha$  and

$\beta$ . We must further specify  $b_z$  to return to the lab frame, but this choice can be made after the evolution operator is computed in the rotating frame. In what follows, I will show that one can systematically find analytical solutions with either  $\alpha$  or  $\beta$  chosen as desired; although one cannot completely specify both  $\alpha$  and  $\beta$  at will (were this the case, one could solve all possible two-state problems analytically), we will see that one still has a large amount of control over the features of the second, unspecified function.

For concreteness, we will suppose that one wishes to fix  $\beta(t)$  at the outset (the formalism can easily be modified to fix  $\alpha(t)$  instead). While we cannot then find analytical solutions for arbitrary  $\alpha$ , there exists a different parametrization of the Hamiltonian in which  $\alpha$  is replaced with a new function,  $\kappa_I(t)$ , such that one can systematically generate an analytical expression for the evolution operator for arbitrary choices of  $\beta$  and  $\kappa_I$ . To see this, first express the rotating-frame evolution operator in terms of some complex  $\kappa(t)$ :

$$v_{11} = e^{-i \int_0^t dt' \beta(t') e^{\kappa(t')}}, \quad v_{21} = -i\eta e^{-i \int_0^t dt' \beta(t') e^{-\kappa(t')}}, \quad (4)$$

with  $\eta = \pm 1$ . This choice of parametrization is motivated by observing that we can combine the two equations in (3) to obtain  $\beta^2 = -(\dot{v}_{11}/v_{11})(\dot{v}_{21}/v_{21})$ , which implies  $\dot{v}_{11}/v_{11} = -i\beta e^{\kappa}$ ,  $\dot{v}_{21}/v_{21} = -i\beta e^{-\kappa}$  for some complex  $\kappa$ . Consistency between Eqs. (4) and (3) requires

$$\alpha(t) = -i\kappa(t) + \eta \frac{\pi}{2} - 2 \int_0^t dt' \beta(t') \sinh \kappa(t'), \quad (5)$$

which should be interpreted as follows: For any choice of a complex  $\kappa(t)$  and real  $\beta(t)$  such that the  $\alpha(t)$  computed from Eq. (5) is real, the evolution operator obtained from Eq. (4) is the exact solution for this  $\alpha$  and  $\beta$ . Writing  $\kappa = \kappa_R + i\kappa_I$  and imposing  $\text{Im}(\alpha) = 0$  determines  $\kappa_R$  in terms of  $\kappa_I$ :  $\kappa_R = -2 \tanh^{-1} \tan(\chi + \pi/4)$ , with

$$\chi(t) \equiv \int_0^t dt' \beta(t') \sin(\kappa_I(t')). \quad (6)$$

From this we find an expression for  $\alpha$  that is real for any choice of  $\kappa_I$ :

$$\alpha = \kappa_I + \eta \frac{\pi}{2} - 2 \int_0^t dt' \beta(t') \cos \kappa_I(t') \cot[2\chi(t')]. \quad (7)$$

While this parametrization has the nice feature that  $\kappa_I$  can be chosen freely, one drawback is that one must then perform the integration in Eq. (6), making it harder to relate the features of  $\kappa_I$  to the driving field  $\dot{\alpha}$ . We can avoid this by specifying  $\chi$  directly, but at the expense of now having to choose functions  $\chi(t)$  that obey  $|\dot{\chi}| \leq |\beta|$ , which arises directly from Eq. (6). In terms of  $\chi$ , the lab-frame evolution operator and driving field  $\dot{\alpha}$  are

$$u_{11} = \cos \chi e^{i\xi_- - i\varphi/2}, \quad u_{21} = i\eta \sin \chi e^{i\xi_+ + i\varphi/2}, \quad (8)$$

$$\xi_{\pm} = \int_0^t dt' \beta \sqrt{1 - \frac{\dot{\chi}^2}{\beta^2}} \csc(2\chi) \pm \frac{1}{2} \sin^{-1} \left( \frac{\dot{\chi}}{\beta} \right) \pm \eta \frac{\pi}{4},$$

$$\dot{\alpha} = \frac{\ddot{\chi} - \dot{\chi}\dot{\beta}/\beta}{\beta \sqrt{1 - \dot{\chi}^2/\beta^2}} - 2\beta \sqrt{1 - \dot{\chi}^2/\beta^2} \cot(2\chi). \quad (9)$$

The initial conditions  $u_{11}(0) = 1$ ,  $u_{21}(0) = 0$  imply  $\chi(0) = 0$ , and  $\dot{\chi}(0) = -\eta\beta(0)$  ensures  $\dot{\alpha}(0)$  is finite. Eqs. (8),(9) embody one of the main results of this paper, as they constitute a general analytic solution of the evolution generated by the Hamiltonian of Eq. (1). The task of finding analytical solutions has been reduced to first picking  $\beta(t)$  at will. One then selects  $\chi(t)$  to produce a desired  $\alpha(t)$  via Eq. (9), fixing the rotating-frame Hamiltonian. Once these choices are made, an analytical expression for the evolution operator follows immediately from Eq. (8). The fact that  $\chi$  directly controls  $|u_{11}|$ ,  $|u_{21}|$  makes this formalism a powerful tool in designing analytical driving fields that execute precise quantum operations or LZSM interferometry, as will be demonstrated shortly with explicit examples. A simple case is  $\chi(t) = -\eta \int_0^t dt' \beta(t')$ , where Eq. (9) gives that  $\dot{\alpha} = 0$ , corresponding to an  $x$ -rotation for any  $\beta$ . Another simple example arises when  $\beta = \chi = 0$ , for which Eq. (8) yields a  $z$ -rotation in the lab frame for any choice of  $b_z$ .

The physical origin of the constraint  $|\dot{\chi}| \leq |\beta|$  lies in the notion of the quantum speed limit,<sup>40–44</sup> which refers to the minimum time it takes a quantum state to evolve to a different state in the Hilbert space due to energy-time uncertainty. Indeed,  $|\dot{\chi}| \leq |\beta|$  implies that the fastest possible evolution from  $\chi(0) = 0$  to  $\chi(T) > 0$  is obtained by choosing  $\chi(t) = \int_0^t dt' |\beta(t')|$ , with the shortest time given by substituting  $t = T$  in this expression and solving for  $T$ . For constant  $\beta = \beta_0 > 0$ , we immediately obtain  $T_{min} = \chi(T)/\beta_0$ , which is the QSL time  $T_{QSL}$  derived for an arbitrary Hilbert space in Ref. 43. We will refer to  $|\dot{\chi}| \leq |\beta|$  as the QSL constraint. The present work leads to a general definition of  $T_{QSL}$ ,  $\chi(T_{QSL}) \equiv \int_0^{T_{QSL}} dt' |\beta(t')|$ , for arbitrary time-dependent two-level systems. Notice that the QSL evolution  $\chi = \pm \int_0^t dt' \beta(t')$  coincides with  $\dot{\alpha} = 0$ , suggesting that the fastest quantum operations are those which tend to minimize  $\dot{\alpha}$ , a tendency that is borne out in the examples given below.

The fact that the QSL appears as a simple condition on  $\chi$  makes the formalism of Eqs. (8),(9) very effective for designing quantum controls that operate near the QSL. To see how this works for a general  $\beta(t)$ , note that a simple way to construct a function  $\chi(t)$  which obeys the QSL constraint is to first find a function which satisfies the constraint in the case where  $\beta(t) = \beta_0$  is a constant. Denoting this latter function by  $\chi_0(t)$  and defining  $B(t) \equiv \int_0^t dt' \beta(t')$ , if we choose  $\chi(t) = \chi_0(B(t)/\beta_0)$ , then  $|\dot{\chi}| = |\beta \chi'_0(B/\beta_0)/\beta_0| \leq |\beta|$  automatically follows. [Note that all the single-axis driving examples of Ref. 35, where the notation there is related to the present by  $q = \cos(2\chi)$ , can be extended to multi-axis solutions using this trick.] Furthermore, if the control field corresponding to  $\chi_0$  operates near the QSL, this will also tend to be the case for the one generated by  $\chi$ . Focusing then on the case  $\beta(t) = \beta_0$ , we can construct controls that

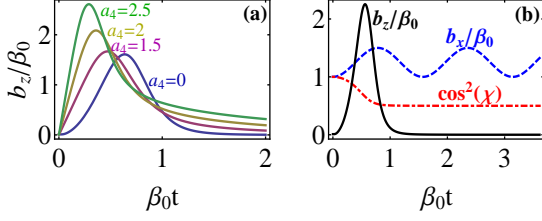


FIG. 1: (Color online) Control field  $b_z$  generated by the  $\chi$  from (a) Eq. (10) with  $b_x = \beta_0$ ,  $\varphi = 0$ ,  $k = 6$ ,  $a_2 = 0$ ,  $a_6 = 4/(\pi\beta_0)$ , (b) Eq. (11) with  $b_x = \beta_0(1 + \sin^2(2\beta_0 t)/2)$ ,  $\varphi = 0$ ,  $k = 6$ ,  $a_2 = a_4 = 0$ ,  $a_6 = 4/\pi$ . A Hadamard gate is achieved for total evolution duration  $T = 3.61/\beta_0$ .

operate near the QSL by choosing a  $\chi(t)$  which contains parameters that can be tuned to values where  $\chi = \pm\pi/4$  and the QSL constraint is saturated.

To illustrate this method, consider the example

$$\chi = -\beta_0 t [1 + (a_2 t)^2 + (a_4 t)^4 + \dots + (a_k t)^k]^{-1/k}, \quad (10)$$

where  $\beta(t) = \beta_0$  and the  $a_i$  are arbitrary constants,  $k$  is an even integer, and  $\eta = 1$ . The QSL constraint is satisfied regardless of how large  $k$  is, so that this  $\chi$  yields an exact solution with arbitrarily many parameters  $a_i$ . We can make the corresponding control field a pulse by setting  $a_k = 4\beta_0/\pi$ , so that  $\chi \rightarrow -\pi/4$  and  $\dot{\chi} \rightarrow 0$  as  $t \rightarrow \infty$ .<sup>52</sup> Examples of these pulses are shown in Fig. 1a. The duration of the pulse approaches  $T_{QSL}$  in the limit  $a_{i < k} \rightarrow 0$ ,  $k \rightarrow \infty$ , as can be seen by observing that  $\chi \rightarrow -\beta_0 t$  in this limit. The substantial amount of tunability in this solution already makes it very attractive for applications in quantum computation such as dynamically corrected gates,<sup>32,34,39</sup> where one tunes the shape of the pulse to perform a specific quantum operation while simultaneously suppressing leading-order errors.

Using the prescription outlined above, we can extend this solution to the case of non-constant  $\beta$ :

$$\chi = -B(t) [1 + (a_2 B(t))^2 + (a_4 B(t))^4 + \dots + (a_k B(t))^k]^{-1/k}. \quad (11)$$

This class of pulses can be used to implement quantum operations by tuning  $b_z(t)$  while keeping the functions  $b_x(t)$  and  $b_y(t)$  fixed. We will demonstrate this by designing a fast  $b_z$  pulse that, together with  $b_x$ , implements a Hadamard gate, a quantum operation that is ubiquitous in the field of quantum information processing and which is equivalent to a  $\pi$ -rotation about  $\hat{x} + \hat{z}$ . First choose  $a_k = 4/\pi$ , which ensures that  $|u_{11}|, |u_{21}| \rightarrow 1/\sqrt{2}$ . Supposing  $\varphi = 0$ , if we let the system evolve for a time  $T$  such that  $\int_0^T dt' \sqrt{\beta^2 - \dot{\chi}^2} \csc(2\chi) = -5\pi/4$ , then the phases of  $u_{11}$  and  $u_{21}$  will also attain their Hadamard values. An example of one such  $b_z$  pulse is shown in Fig. 1b for the case where  $b_x$  is a particular oscillatory function. From the figure, it is evident that the  $b_z$  pulse quickly sets the magnitudes of  $u_{11}$  and  $u_{21}$ , while the remainder of the

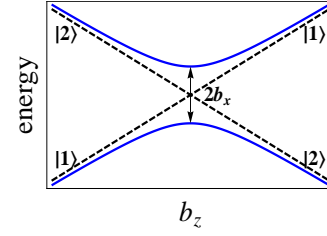


FIG. 2: (Color online) Diabatic and adiabatic energy levels as a function of  $b_z$ .

evolution with  $b_z \approx 0$  allows their phases to accumulate. As before, the duration of the pulse approaches  $T_{QSL}$  as  $a_{i < k} \rightarrow 0$ ,  $k \rightarrow \infty$ . This example illustrates how one can use this formalism to design analytical single or multi-axis quantum controls near the QSL in the presence of additional continuous-driving fields.

In addition to designing control pulses that implement fast quantum gates, the present formalism is also natural for designing driving fields that perform controlled LZSM interferometry and non-adiabatic rapid passage,<sup>53</sup> phenomena which have many applications in quantum control,<sup>8–11</sup> state preparation<sup>12,13</sup> and qubit readout.<sup>36–38</sup> The LZSM problem is generally setup as follows. Define the eigenstates of  $\sigma_z$  to be  $|1\rangle$  and  $|2\rangle$  and set  $\varphi = 0$  so that  $\beta = b_x$  and  $\dot{\alpha} = 2b_z$ ; when  $|b_z| \gg |b_x|$ , these states are approximate energy eigenstates. Nonzero  $b_x$  produces an anti-crossing with an energy gap of  $2b_x$  (see Fig. 2) which may be time-dependent. Now suppose that we drive  $b_z$  through the anti-crossing, starting from some large negative value at  $t = 0$  up to a large positive value at  $t = T$ . Assuming that the system is initially prepared in state  $|1\rangle$  at time  $t = 0$ , the probability  $P_2(T)$  for the system to be in state  $|2\rangle$  at time  $t = T$  is

$$P_2(T) = |u_{21}(T)|^2 = \sin^2[\chi(T)]. \quad (12)$$

The fact that this depends only on  $\chi(T)$  demonstrates the suitability of the present formalism to the LZSM problem. If we choose  $\chi$  such that  $\chi(0) = 0$  and  $\chi(T) = 0$ , then we achieve a perfect LZSM transition: the system is driven through the anti-crossing and returns to state  $|1\rangle$  with probability 1. On the other hand, we may choose  $\chi(T) = \pi/2$ , in which case the system undergoes NARP and ends up in state  $|2\rangle$  after being driven through the anti-crossing. Other values of  $\chi(T)$  lead to superpositions of  $|1\rangle$  and  $|2\rangle$ . We may also consider LZSM interferometry, where the system is driven through the anti-crossing periodically, and the resulting time-averaged probabilities to be in state  $|1\rangle$  or  $|2\rangle$  after many periods is again largely determined by  $\chi(T)$ , as we will see. In choosing a  $\chi(t)$  for the LZSM problem, we must impose appropriate initial conditions. For simplicity, we focus on the case  $b_z(0) = -\infty$ ,  $b_z(T) = \infty$ , for which we need  $\ddot{\chi}(0) < 0$ ,  $\ddot{\chi}(T) > 0$ ; the analysis can be extended straightforwardly to the case where  $b_z$  is finite at  $t = 0, T$ .

For constant  $b_x$ , a simple example which satisfies these

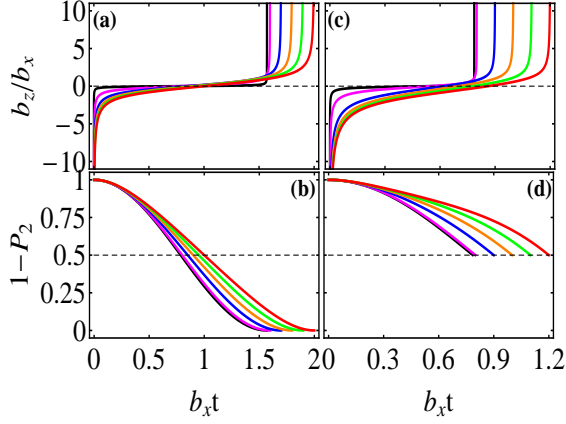


FIG. 3: (Color online) Control field and NARP probability from Eqs. (13),(12),(9) with (a),(b)  $\chi(T) = \pi/2$  and  $b_x T = \pi/2, 1.6, 1.7, 1.8, 1.9, 2$  and (c),(d)  $\chi(T) = \pi/4$  and  $b_x T = \pi/4, 0.8, 0.9, 1, 1.1, 1.2$ .

boundary conditions and the QSL constraint is

$$\chi = b_x t - \frac{ab_x T}{2} t^2 + \frac{ab_x}{3} t^3, \quad (13)$$

where choosing  $0 \leq a \leq 16/(3T^2)$  ensures that the  $b_z = \dot{\chi}/2$  from Eq. (8) remains finite in the interval  $t \in (0, T)$ , and  $\eta = -1$ . This  $\chi$  saturates the QSL constraint when  $a = 0$ , implying that the corresponding  $b_z$  will implement near-QSL evolution for small  $a$ . To achieve a target evolution characterized by  $\chi(T)$ , set  $t = T$  in Eq. (13) and solve for  $a$ :  $a(T) = 6[b_x T - \chi(T)]/(b_x T^3)$ . Plugging this expression into Eqs. (13) and (9) yields a family of driving fields  $b_z$  parametrized by  $T$  that will achieve the desired evolution for any value of  $\chi(T) \in (0, \pi/2]$ ; some of these solutions along with the corresponding NARP probabilities are shown in Fig. 3. The restrictions on  $a$  become restrictions on  $T$  through the expression for  $a(T)$ :  $\chi(T) \leq b_x T \leq 9\chi(T)$ ; while the upper bound stems from the particular choice of  $\chi$ , Eq. (13), the lower bound is the familiar, universal QSL and gives rise to the step-like curves in Figs. 3a and 3c. These curves reveal that the desired LZSM transition is achieved as quickly as possible by first driving  $b_z$  to zero very rapidly, allowing the system to evolve under  $\beta(t)$  for a time  $T \lesssim T_{QSL}$ , and finally driving  $b_z$  quickly up to its final, large value. In addition to performing NARP, these near-QSL driving fields could be important for LZSM-based generation of entanglement in superconducting qubits,<sup>10</sup> where fidelities are often limited by short relaxation times.

In the context of LZSM interferometry, the formalism of Eqs. (8),(9) yields an exact formula for LZSM interference patterns. To show this, we begin by constructing a periodic driving field  $b_z$  of period  $2T$  in which the first half of one period is described by some choice of  $\chi(t)$  and the second half by  $\chi_2(t) = \chi(2T - t)$ , corresponding to  $b_z$  retracing its path. Using Eq. (8), it is simple to find

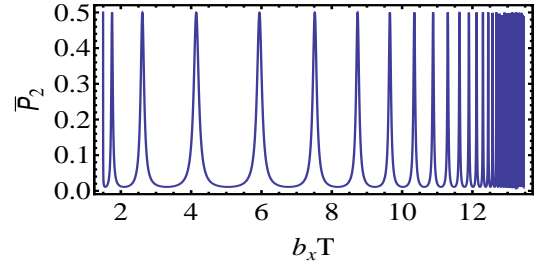


FIG. 4: (Color online) Time-averaged excited state probability  $\bar{P}_2$  from Eq. (15) with periodic driving field generated by Eq. (13) with  $\chi(T) = \pi/2.1$  as a function of the drive half-period  $T$ .

the evolution after one full period:

$$\begin{aligned} u_{11}(2T) &= e^{2i\xi_0(T)} \cos(2\chi(T)), \\ u_{21}(2T) &= -i \sin(2\chi(T)), \end{aligned} \quad (14)$$

where  $\xi_0 \equiv (\xi_+ + \xi_-)/2$  and we have assumed that  $\dot{\chi}(T) = -\eta\beta(T)$  for simplicity. From this expression, it is then straightforward to compute the time-averaged probability of being in state  $|2\rangle$  (which is the excited state since each period begins and ends to the left of the anti-crossing) after many periods:

$$\bar{P}_2 = [2 + 2 \cot^2(2\chi(T)) \sin^2(2\xi_0(T))]^{-1}. \quad (15)$$

Thus, we see that the present formalism readily produces a general, exact, analytic formula for  $\bar{P}_2$ , whereas analytic expressions for this important quantity typically require several approximations.<sup>5</sup> It is clear that this function takes values in the range  $[0, 1/2]$  and that  $\bar{P}_2 = 1/2$  for  $\chi(T) = \pi/4$ , while  $\bar{P}_2 = 0$  for  $\chi(T) = \pi/2$ . This is to be expected since  $\chi(T) = \pi/4$  corresponds to a perfect 50-50 “beam splitter”, while  $\chi(T) = \pi/2$  ensures the system remains in the ground state after every sweep through the anti-crossing. In the context of charge qubits in which an electron inhabits a tunnel-coupled double quantum dot and states  $|1\rangle$  and  $|2\rangle$  correspond to the electron being in either the left dot or the right, the case  $\bar{P}_2 = 0$  can be interpreted as complete coherent destruction of tunneling<sup>45,46</sup> since the electron is then fully localized in one of the dots. For more generic values of  $\chi(T)$ ,  $\bar{P}_2$  is modulated by the phase  $\xi_0(T)$ , which can be tuned independently of  $\chi(T)$  and which can produce interference fringes as control parameters are adjusted. This is illustrated in Fig. 4 for the example from Eq. (13) with  $\chi(T) = \pi/2.1$ . As the half-period drive time  $T$  is increased, an interference pattern emerges in which the peaks of the pattern sharpen and eventually disappear as  $\chi(T)$  approaches  $\pi/2$ .<sup>54</sup> Interestingly, Fig. 4 reveals that there is a peak at the QSL time  $T = T_{QSL} = \pi/(2.1b_x)$ ; this is generally the case since at the QSL,  $|\dot{\chi}| = |\beta|$ , so that  $\xi_0(T_{QSL}) = 0$ . This leads to the surprising conclusion that if one wishes to trap the system in state  $|2\rangle$  by choosing  $\chi(T) = \pi/2$ , then driving very close to the

QSL may not be ideal since small deviations away from  $\chi(T) = \pi/2$  will produce the peak at  $T = T_{QSL}$  and, hence, large uncertainty in the state of the system.

In conclusion, a general formalism for deriving exactly solvable time-dependent two-level quantum systems was presented. This formalism can vastly increase the number of known exact solutions for physical Hamiltonians, as has been demonstrated with explicit, important ex-

amples. These examples clearly show that this theory is a powerful tool in the design of control pulses both for quantum computation and for precise Landau-Zener interferometry near the quantum speed limit.

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